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SOME REMARKS ON METAPLECTIC CUSP FORMS AND THE CORRESPONDENCES OF SHIMURA AND WALDSPURGER

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ABSTRACT
The simple relation between representations of the covering groups of SL_2 and GL_2 makes it possible to fuse and extend the recent metaplectic results of Shimura, Waldspurger, Flicker, and ourselves. By giving a new (purely local and L-function theoretic) treatment of the Waldspurger-Shintani correspondence, we also simplify some of Waldspurger's original results.

Introduction

In 1973, G. Shimura used L-functions to prove the existence of an intriguing correspondence between holomorphic cusp forms of half-integral and even-integral weight ([12]). Shortly afterwards, T. Shintani and S. Niwa gave a more direct construction of Shimura's correspondence using theta-series (or a Weil representation) attached to a quadratic form in 3-variables (see [13] and [10]). More recently, all these works have been generalized to the context of representations and adeles. First the present authors ([3]) extended Shimura's L-function theory to arbitrary representations of the metaplectic cover of GL_2. Then Flicker ([1]) exploited the Selberg trace formula to obtain still stronger results for n-fold covers of GL_2. Finally J.-L. Waldspurger in [16] gave a detailed and adelic analysis of Shintani's construction, thereby obtaining surprising new results for the L-functions of GL_2 as well.

The remarks of the present paper are intended to complement all the aforementioned works. Our purpose is to make completely explicit the relation between the correspondence of Shimura and Shintani–Waldspurger and to explain how Flicker's work makes possible shorter proofs of some of Waldspurger's most important results.

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Shimura's correspondence (or rather its generalization in [3] or [1]) deals with a metaplectic cover of GL_2, whereas the Shintani–Waldspurger correspondence deals with the metaplectic cover of SL_2. Thus it is necessary to first make a comparison of the theory of representations of these covering groups. This analysis is carried out in §1. Our observation that the process of induction gives a simple correspondence between the irreducible representations of these groups makes it possible to translate Flicker's strong multiplicity one result for the covering groups of GL_2 ([1], p. 180) into a similar strong multiplicity one result for the metaplectic cover of $SL_2(\mathbf{A})$ (Theorem 1.4 of the present paper). This metaplectic result was first conjectured in [17] and is now proved by completely different methods in [18]; it is to be contrasted to the situation for SL_2, where strong multiplicity one fails (cf. [9] and §(1.4) below).

In §2 we describe how the correspondences of Shimura and Shintani–Waldspurger are related (The "Key Diagram" (2.0)). Our approach is purely local: we reformulate Waldspurger's correspondence in terms of the local Shimura type zeta-functions analyzed in [3]. As a corollary, we obtain a simple derivation of the ε-factor assertions of §6 of [3]. Global consequences of the "Key Diagram", again using [1], are described in §3.

Finally, in §4, we introduce a notion of "near-equivalence" (or "L-indistinguishability") which seems appropriate for cuspidal representations of the metaplectic cover of $SL_2(\mathbf{A})$, and we describe the resulting "L-packets" following [18].

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Notation and preliminaries

Throughout, we follow the notation and definitions of [3] and [4], to which the reader is referred for details. In particular,

F is a local or global field,

$G = GL_2$ (regarded as an algebraic group over F),

$S = SL_2$,

and $\bar{G}$ is the 2-fold covering group of G described in [3] as a group of pairs $\{(g,\xi): g \in G, \xi \in \{\pm 1\}\}$. If $H \subset G$ is any subgroup, then $\bar{H}$ denotes the inverse image of H under the natural projection of $\bar{G}$ onto G; if the extension $\bar{G}$ "splits" over H, then $\bar{H}$ is the direct product of $\{\pm 1\}$ with a subgroup of $\bar{G}$ which we again denote by H.

For the purposes of the present paper, we also need to introduce an intermediate group $S \subset G^* \subset G$ as follows: if $\mathbf{G}_m$ denotes the "multiplicative group" GL_1, then

$$G^* = \{g \in G : \det g \in \mathbf{G}_m^2\}.$$

In particular, when F is a local field,

$$G_F^* = \{g \in \mathrm{GL}_2(F) : \det g \in (F^*)^2\}.$$

Finally, we recall that a representation of $\bar{G}$ (or any subgroup $\bar{H}$ thereof) is called *genuine* if its restriction to $\{\pm 1\}$ transforms according to the non-trivial character of $\{\pm 1\}$.

§1. Induced representations and the strong multiplicity one theorem for $\bar{S}_{\mathbf{A}}$

Suppose $\bar{\pi}$ is a genuine irreducible automorphic cuspidal representation of $\bar{G}_{\mathbf{A}}$. Our purpose in this section is to show that $\bar{\pi}$ is essentially induced from an irreducible cuspidal representation of $\bar{S}_{\mathbf{A}}$ contained in the restriction of $\bar{\pi}$ to $\bar{S}_{\mathbf{A}}$. More precisely, we shall deal with the intermediate subgroup

$$\bar{G}_{\mathbf{A}}^* = \{(g, \xi) : \det g \in (\mathbf{A}^\times)^2\}.$$

We show that induction gives rise to a one-one correspondence

$$\bar{\pi} \to \bar{\pi}_*$$

between the genuine automorphic representations of $\bar{G}_{\mathbf{A}}$ and $\bar{G}_{\mathbf{A}}^*$; this bijection is already implicit in [1], where the characters of genuine $\bar{\pi}$ on $\bar{G}$ were shown to be supported on $\bar{G}^*$.

(1.1) *Local Theory*

Let F denote a non-archimedian local field of odd residual characteristic, and $\bar{\sigma}$ a genuine irreducible admissible representation of $\bar{S}$. If $g \in G$, the equivalence class of the conjugate representation

$$\bar{\sigma}^g(\bar{s}) = \bar{\sigma}((g,1)\bar{s}(g,1)^{-1}), \qquad \bar{s} \in \bar{S}, \tag{1.1.1}$$

depends only on $\det(g)$. This is because $\bar{G}$ is the semi-direct product of $\bar{S}$ with the group

$$\left\{ \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} : a \in F^\times \right\}$$

(think of a as $\det(g)$; cf. [4]). The center of $\bar{S}$ is the four-element (Klein 4-group) $\{(\pm I, \xi) : \xi = \pm 1\}$, the intersection of $\bar{S}$ with

$$\bar{Z} = \left\{ \left(\begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} ; \xi \right) : \alpha \in F^x, \xi = \pm 1 \right\}.$$

Recall the group $\bar{G}^* = \{(g, \xi) : g \in G, \det g \in (F^x)^2\}$. Its center is $\bar{Z}$.

Let μ be any genuine character of $\bar{Z}$ whose restriction to $\bar{S} \cap \bar{Z}$ agrees with the central character of $\bar{\sigma}$. Since $\bar{G}^* = \bar{S} \times \bar{Z}$, the formula

$$\bar{\sigma} \times \mu(\bar{s}\bar{z}) = \bar{\sigma}(\bar{s})\mu(\bar{z}) \tag{1.1.2}$$

determines a well-defined irreducible admissible representation of $\bar{G}^*$ whose restriction to $\bar{S}$ (resp. $\bar{Z}$) is $\bar{\sigma}$ (resp. μ).

Note that the central character of $\bar{\sigma} \times \mu$ is μ, since $\bar{Z}$ is the center of $\bar{G}^*$. However, the central character of the conjugate representation $(\bar{\sigma} \times \mu)^g$ is μ^a, where $a = \det g$ and

$$\mu^a(\bar{z}) = \mu\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \bar{z} \begin{bmatrix} a^{-1} & 0 \\ 0 & 1 \end{bmatrix} \right) = (a, z)\mu(\bar{z}) \quad \text{for } \bar{z} = \left(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix}, \xi \right). \tag{1.1.3}$$

(Here (a, z) denotes the Hilbert symbol of the scalars a and z in F^x.) Thus $(\bar{\sigma} \times \mu)^g$ is equivalent to $\bar{\sigma} \times \mu$ if and only if $\det g \in (F^x)^2$, and so from Mackey's theory we obtain:

Proposition 1.1.4. *The induced representation*

$$\bar{\pi} = \mathrm{Ind}(\bar{\sigma} \times \mu, \bar{G}^*, \bar{G})$$

is irreducible, and its restriction to $\bar{G}^*$ is the direct sum of the representations $(\bar{\sigma} \times \mu)^{\left[\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right]}$, as a runs through representatives of $(F^x)^2 \backslash F^x$ in F^x.

Remark. The analogue of this proposition for GL_2 and G^* (in place of $\bar{G}$ and $\bar{G}^*$) is false; in particular, if σ is a generic principal series representation of SL_2, $(\sigma \times \mu)^g$ is *always* equivalent to $\sigma \times \mu$.

Returning to the metaplectic group, let us suppose conversely that $\bar{\pi}$ is an irreducible admissible genuine representation of $\bar{G}$ in some space $V_{\bar{\pi}}$, and the central character of $\bar{\pi}$ is $\omega_{\bar{\pi}}$, i.e., $\omega_{\bar{\pi}}$ is a genuine character of $\bar{Z}^2$, and

$$\bar{\pi}(\bar{z}) = \omega_{\bar{\pi}}(\bar{z})I \qquad \forall \bar{z} \in \bar{Z}^2. \tag{1.1.5}$$

Let $\Omega_{\bar{\pi}}$ denote the (finite) set of characters of $\bar{Z}$ whose restriction to $\bar{Z}^2$ is $\omega_{\bar{\pi}}$. Then the natural action of $\bar{Z}$ on $V_{\bar{\pi}}$ decomposes $V_{\bar{\pi}}$ into the direct sum of eigenspaces

$$(1.1.6) \qquad V^{\mu}=\{v \in V_{\bar{\pi}} : \bar{\pi}(\bar{z})v=\mu(\bar{z})v \quad \forall \bar{z} \in \bar{Z}\},$$

one for each μ in $\Omega_{\bar{\pi}}$.

Since $\bar{G}^*$ commutes with $\bar{Z}$, each V^{μ} is invariant for the action of $\bar{G}^*$ through $\bar{\pi}$. The resulting $\bar{G}^*$-module V^{μ} obviously has central character μ, so V^{μ} is equivalent to $V^{\mu'}$ if and only if $\mu=\mu'$. Moreover, the identity (1.1.3) implies

$$\bar{\pi}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} V^{\mu}=V^{\mu^a}.$$

Thus

$$(1.1.7) \qquad V_{\bar{\pi}} \mid \bar{G}^*=\bigoplus_a V^{\mu^a},$$

the sum extending over *all* representatives a in F^x for the coset space $F^x/(F^x)^2$.

Indeed, the map $\mu \to \mu_0\mu^{-1}$, with μ_0 fixed in $\Omega_{\bar{\pi}}$, defines a bijection from $\Omega_{\bar{\pi}}$ to the set of characters of $F^x/(F^x)^2$. So since the Hilbert symbol $(.,.)$ identifies $F^x/(F^x)^2$ as its own Pontryagin dual, it follows from (1.1.3) that the set

$$\left\{\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} : a \in F^x\right\}$$

acts transitively on the collection of V^{μ}'s.

REMARK. Let π^{μ} denote the representation of $\bar{G}^*$ realized in the eigenspace V^{μ}. If $g \in G$, and $\det(g)=a$, then the conjugate representation $(\pi^{\mu})^g$ acts in V^{μ^a} and is equivalent to π^{μ^a}. In particular, (1.1.7) may be rewritten in the form

$$(1.1.7)^* \qquad V_{\bar{\pi}} \mid \bar{G}=\bigoplus_a (V^{\mu})^{\left[\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right]}.$$

Frequently, we shall not distinguish between a representation and the space in which it acts. Thus, from (1.1.7)* we obtain:

PROPOSITION 1.1.8. *For any μ in $\Omega_{\bar{\pi}}$, the $\bar{G}^*$-module V^{μ} is irreducible, and*

$$\bar{\pi}=\mathrm{Ind}(V^{\mu}, \bar{G}^*, \bar{G}).$$

PROOF. (1.1.7) implies $\bar{\pi}$ is induced from any V^{μ}, and since $\bar{\pi}$ was assumed irreducible, V^{μ} must be also. Note that the transitive action of $\bar{G}/\bar{G}^*$ on the V^{μ}'s implies that the restrictions of two representations $\bar{\pi}_1$ and $\bar{\pi}_2$ to $\bar{G}^*$ either coincide or are completely disjoint.

CONCLUDING REMARK. Suppose $\bar{\pi}$ is an unramified representation of $\bar{G}$, and y_0 is a vector in $V_{\bar{\pi}}$ fixed by $\mathrm{GL}_2(O_F)$. Then in the decomposition (1.1.7) no eigenspace V^{μ} contains y_0, i.e., no V^{μ} contains a $\mathrm{GL}_2(O_F)$-invariant vector. Indeed, suppose one did, say V^{μ_0}. For any a in $F^{\times}$, the vector $\bar{\pi}\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} y_0$ belongs to V^{μ^a}, so for α any non-square *unit*, we would have another $\mathrm{GL}_2(O_F)$-invariant vector, namely $\bar{\pi}\left(\left[\begin{smallmatrix} \alpha & 0 \\ 0 & 1\end{smallmatrix}\right]\right) y_0$, in $V^{\mu_0^{\alpha}}$, a contradiction since V^{μ_0} and $V^{\mu_0^{\alpha}}$ are disjoint, and the space of $\mathrm{GL}_2(O_F)$-invariant vectors in $V_{\bar{\pi}}$ is one dimensional.

(1.2) *Global Theory*

Once again F is an A-field and $\bar{\pi} = \bigotimes \bar{\pi}_v$ is an irreducible unitary genuine representation of $\bar{G}_{\mathbf{A}}$ in some space $V_{\bar{\pi}}$. As in the local theory, we shall show that $(\bar{\pi}, V_{\pi})$ is induced from an irreducible representation $\bar{\pi}^{\mu}$ of $\bar{G}^*_{\mathbf{A}}$, with μ a character of $\bar{Z}_{\mathbf{A}}$ whose restriction to $\bar{Z}^2_{\mathbf{A}}$ is $\omega_{\bar{\pi}}$. The new twist, however, is that $\bar{\pi}^{\mu}$ will no longer be a subrepresentation of $\bar{\pi}$ and the number of possible π^{μ} is not countable.

Let $\Omega_{\bar{\pi}}$ denote the set of characters of $\bar{Z}_{\mathbf{A}}$ extending $\omega_{\bar{\pi}}$ on $\bar{Z}^2_{\mathbf{A}}$, and fix $\mu = \bigotimes \mu_v$ in $\Omega_{\bar{\pi}}$. For *any* place v of f, let $\bar{\pi}^{\mu}_v$ denote the irreducible representation of $\bar{G}^*_v$ acting in the eigenspace V^{μ_v}; cf. (1.1.6) and (1.1.7). Let S denote the finite set of places of F such that

(i) v is finite and "odd" outside S;

(ii) $\bar{\pi}_v$ and μ_v are unramified outside S.

For each v *outside* S, $\bar{\pi}_v$ has a vector invariant with respect to $\mathrm{GL}_2(O_v)$ but in the sense of the Concluding Remark of §1.1, $\bar{\pi}^{\mu}_v$ does not. Set

$$K^*_v = \{g \in \mathrm{GL}_2(O_v) : \det g \in (F^{\times}_v)^2\}.$$

Then for each $v \notin S$, the space V^{μ_v} will contain a vector invariant for K^*_v, and the (restricted) tensor product

$$\bar{\pi}^{\mu} = \bigotimes_v \bar{\pi}^{\mu}_v$$

can be defined.

The representation $\bar{\pi}^{\mu}$ determines an *irreducible* representation of $\bar{G}^*_{\mathbf{A}}$ since each local component $\bar{\pi}^{\mu}_v$ is an irreducible representation of $\bar{G}^*_v$. However, $\bar{\pi}^{\mu}$ is not a subrepresentation of $\bar{\pi}$ (regarded as a $\bar{G}^*_{\mathbf{A}}$-module) since no vector in the space of $\bar{\pi}^{\mu}$ is fixed by $\mathrm{GL}_2(O_v)$ for almost every v (whereas every vector in $V_{\bar{\pi}}$ is so fixed).

PROPOSITION 1.2.1. *For each character μ in $\Omega_{\bar{\pi}}$ the representation* $\mathrm{Ind}(\bar{\pi}^{\mu}, \bar{G}^*_{\mathbf{A}}, \bar{G}_{\mathbf{A}})$ *is equivalent to* $\bar{\pi}$.

PROOF. Follows from the local results.

When $\bar{\pi}$ happens to be automorphic cuspidal, we can refine Proposition 1.2.1 as follows. Suppose $\bar{\pi}$ is realizable in some subspace of $V_{\bar{\pi}}$ of square-integrable cusp forms on $\bar{G}_{\mathbf{A}}$. Let $V_{\bar{\pi}_*}$ denote the space of restrictions of functions in $V_{\bar{\pi}}$ from $\bar{G}_{\mathbf{A}}$ to $\bar{G}^*_{\mathbf{A}}$, and $\bar{\pi}_*$ the resulting representation of $\bar{G}^*_{\mathbf{A}}$ in $V_{\bar{\pi}_*}$ given by right translation.

Consider the *compact* group

$$C = \bar{Z}_{\mathbf{A}}/\bar{Z}^2_{\mathbf{A}}Z_F.$$

Since $\bar{Z}_{\mathbf{A}}$ commutes with the action of $\bar{G}^*_{\mathbf{A}}$, and each f in $V_{\bar{\pi}_*}$ is Z_F invariant, the compact group C acts naturally in the space $V_{\bar{\pi}_*}$. The result is a decomposition

$$V_{\bar{\pi}_*} = \bigoplus_{\mu} V_{\bar{\pi}^{\mu}_*} \tag{1.2.2}$$

corresponding to the (countable) set of characters μ of $\bar{Z}_{\mathbf{A}}/Z_F$ whose restriction to $\bar{Z}^2_{\mathbf{A}}$ is $\omega_{\bar{\pi}}$, i.e., the "automorphic" μ in $\Omega_{\bar{\pi}}$. Each subspace $V_{\bar{\pi}^{\mu}_*}$ is invariant under the action of $\bar{G}^*_{\mathbf{A}}$, and an eigenspace for the action of $\bar{Z}_{\mathbf{A}}$.

PROPOSITION 1.2.3. *Let $\bar{\pi}^{\mu}_*$ denote the natural representation of $\bar{G}^*_{\mathbf{A}}$ in $V_{\bar{\pi}^{\mu}_*}$, μ as above. Then $\bar{\pi}^{\mu}_*$ is an irreducible automorphic cuspidal representation of $\bar{G}^*_{\mathbf{A}}$, and the induced representation* $\mathrm{Ind}(\bar{\pi}^{\mu}_*, \bar{G}^*_{\mathbf{A}}, \bar{G}_{\mathbf{A}})$ *is equivalent to $\bar{\pi}$. Moreover, the decomposition* (1.2.2) *may be rewritten in the form*

$$V_{\bar{\pi}_*} = \bigoplus_{a} (V^{\mu_0}_{\bar{\pi}_*})^a \tag{1.2.4}$$

with μ_0 a fixed automorphic character in $\Omega_{\bar{\pi}}$, and the sum extending over all a in F^x.

PROOF. Most of this follows from the local theory. What requires proof is the decomposition (1.2.4). It amounts to the assertion that all "automorphic" μ in $\Omega_{\bar{\pi}}$ are of the form μ_0^a for some a in F^x. Indeed

$$\pi\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} V^{\mu}_{\bar{\pi}_*} = (V^{\mu}_{\bar{\pi}_*})^a = V^{\mu^a}_{\bar{\pi}_*},$$

where $(V^{\mu^a}_{\bar{\pi}_*})^a$ denotes the $\bar{G}^*_{\mathbf{A}}$-module $V_{\bar{\pi}^{\mu}_*}$ conjugated by the element $[\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}]$.

Identify $\bar{Z}_{\mathbf{A}}/\bar{Z}^2_{\mathbf{A}}$ with $\mathbf{A}^x/(\mathbf{A}^x)^2$. If μ_0 in $\Omega_{\bar{\pi}}$ is trivial on Z_F, the map $\mu \to \mu_0\mu^{-1}$ establishes a bijection between the "automorphic" μ in $\Omega_{\bar{\pi}}$ and the characters of $\mathbf{A}^x/(\mathbf{A}^x)^2F^x$. But the Pontryagin dual of $\mathbf{A}^x/(\mathbf{A}^x)^2F^x$ is exactly $F^x/(F^x)^2$, the pairing being provided by the global Hilbert symbol. So if μ is a genuine character of $\bar{Z}_{\mathbf{A}}/Z_F$ whose restriction to $\bar{Z}_{\mathbf{A}}^2$ is $\omega_{\bar{\pi}}$, there exists some a in $F^\times$ such that

$$\mu(\bar{z}) = \mu_0^a(\bar{z}) = (a, z)\mu_0(\bar{z})$$

for all $\bar{z} = \left(\left[\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}\right], \xi\right)$ in $\bar{Z}_{\mathbf{A}}$.

LEMMA 1.2.5. (cf. pp. 768–771 of [9]). *Suppose $\bar{\pi}_*$ is any irreducible genuine representation of $\bar{G}_{\mathbf{A}}^*$ realizable in the space $V_{\bar{\pi}_*}$ of automorphic cusp forms on $\bar{G}_{\mathbf{A}}^*$. Let* $\pi = \mathrm{Ind}(V_{\bar{\pi}_*}, \bar{G}_{\mathbf{A}}^*, \bar{G}_{\mathbf{A}})$. *Then $\bar{\pi}$ is an irreducible automorphic representation of $\bar{G}_{\mathbf{A}}$.*

PROOF. By the local theory, $\bar{\pi}$ is irreducible. To see that it is automorphic, let $l: V_{\bar{\pi}_*} \to \mathbf{C}$ be the non-zero linear functional obtained by evaluating f in $V_{\bar{\pi}_*}$ at the identity. Since $V_{\bar{\pi}_*}$ is a space of automorphic forms, the functional l is invariant for all γ in G_F^*. To show that $\bar{\pi}$ is automorphic cuspidal, we construct an embedding of $V_{\bar{\pi}}$ into the space of cusp forms on $\bar{G}_{\mathbf{A}}$.

By definition, we may take $\bar{\pi}$ to operate by right translation in the space of functions $F: \bar{G}_{\mathbf{A}} \to V_{\bar{\pi}_*}$ which are compactly supported modulo $\bar{G}_{\mathbf{A}}^*$ and such that $F(\bar{s}\bar{g}) = \bar{\pi}_*(\bar{s})F(\bar{g})$ for all $\bar{s}$ in $\bar{G}_{\mathbf{A}}^*$, $\bar{g}$ in $\bar{G}_{\mathbf{A}}$. Define a functional L on the space of $\bar{\pi}$ by

$$L(F) = \sum_{\delta \in F^\times/(F^\times)^2} l\left(F\begin{bmatrix} \delta & 0 \\ 0 & 1 \end{bmatrix}\right).$$

This sum converges because F has compact support modulo $\bar{G}_{\mathbf{A}}^*$, and the functional it defines is clearly invariant for all γ in G_F. Thus the fact that $\bar{\pi}$ is automorphic cuspidal follows from the fact that $\bar{\pi}_*$ is automorphic cuspidal, the embedding of $V_{\bar{\pi}}$ into the space of $\omega_{\bar{\pi}}$-cusp forms on $\bar{G}_{\mathbf{A}}$ being given by the map

$$F \to \phi_F(g) = L(\bar{\pi}(\bar{g})F).$$

Summing up, we have:

PROPOSITION 1.2.6. (i) *Each automorphic cuspidal representation $\bar{\pi}$ of $\bar{G}_{\mathbf{A}}$ is induced from an automorphic cuspidal representation $\bar{\pi}_*$ of $\bar{G}_{\mathbf{A}}^*$, and all such representations of $\bar{G}_{\mathbf{A}}^*$ thus arise.*

(ii) *The "automorphic restriction" of* $\bar{\pi} = \mathrm{Ind}(\bar{\pi}_*, \bar{G}_{\mathbf{A}}^*, \bar{G}_{\mathbf{A}})$ *to* $(\bar{G}_{\mathbf{A}}^*, V_{\bar{\pi}_*})$ *(automorphic cusp forms on $\bar{G}_{\mathbf{A}}^*$) contains precisely the representation* $\bar{\pi}_*^{\left[\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right]}$, *$a$ running through $F^\times$.*

(1.3) *Some Remarks about $\bar{S}_{\mathbf{A}}$*

Recall that $\bar{G}_{\mathbf{A}}^* = \bar{S}_{\mathbf{A}} \times \bar{Z}_{\mathbf{A}}$, with $\bar{S}_{\mathbf{A}} \cap \bar{Z}_{\mathbf{A}}$ equal to the center of $\bar{S}_{\mathbf{A}}$. Thus every (genuine) irreducible unitary representation $\bar{\pi}_*$ of $\bar{G}_{\mathbf{A}}^*$ has the form

$$\bar{\pi}_* = \bar{\sigma} \times \mu$$

where $\bar{\sigma}$ is a genuine irreducible unitary representation of $\bar{S}_{\mathbf{A}}$, μ is a genuine character of $\bar{Z}_{\mathbf{A}}$ which agrees with $\bar{\sigma}$ on $\bar{S}_{\mathbf{A}} \cap \bar{Z}_{\mathbf{A}}$, and $\bar{\sigma} \times \mu$ is defined by the formula

$$\bar{\sigma} \times \mu(\bar{s}\bar{z}) = \bar{\sigma}(s)\mu(\bar{z}).$$

Note $\bar{\sigma} \times \mu$ is automorphic iff both $\bar{\sigma}$ and μ are.

PROPOSITION 1.3.1. *Suppose $\bar{\pi}$ is an irreducible genuine cuspidal representation of $\bar{G}_{\mathbf{A}}$ and μ is a character of $\bar{Z}_{\mathbf{A}}/Z_F$ whose restriction to $\bar{Z}^2_{\mathbf{A}}$ is $\omega_{\bar{\pi}}$. Then the representation $\bar{\pi}^{\mu}_*$ introduced in Proposition* 1.2.3 *is of the form $\bar{\sigma} \times \mu$, for some irreducible cuspidal representation $\bar{\sigma}$ of $\bar{S}_{\mathbf{A}}$, and the map*

$$\bar{\pi} \to \mathrm{Res}_{\mu}(\bar{\pi}) = \bar{\sigma}$$

is well-defined.

Now suppose $\bar{\sigma}$ is any genuine automorphic cuspidal representation of $\bar{S}_{\mathbf{A}}$. Extending $\bar{\sigma}$ up to $\bar{G}^*_{\mathbf{A}}$ by the formula above, and inducing up to $\bar{G}_{\mathbf{A}}$, it follows from §1.2 that the multiplicity of $\bar{\sigma}$ in the space of cusp forms on $\bar{S}_{\mathbf{A}}$ cannot exceed the multiplicity of the corresponding cuspidal representation of $\bar{G}_{\mathbf{A}}$ in its space of cusp forms. But Flicker has shown that this latter multiplicity is one (p. 180 of [1]). Thus we have:

PROPOSITION 1.3.2. *Suppose $\bar{\sigma}$ is a genuine automorphic cuspidal representation of $\bar{S}_{\mathbf{A}}$, and $m(\bar{\sigma})$ denotes its multiplicity in the space of cusp forms on $\bar{S}_{\mathbf{A}}$. Then*

$$m(\bar{\sigma}) = 0 \quad \textit{or} \quad 1.$$

REMARK. An ingenious and involved proof of this multiplicity one result — completely independent of Flicker's result for $\bar{G}_{\mathbf{A}}$ — was the subject matter of [16].

PROPOSITION 1.3.3. *Suppose $\bar{\sigma}$ is an automorphic cuspidal genuine representation of $\bar{S}_{\mathbf{A}}$. For each $\bar{g}$ in $\bar{G}_{\mathbf{A}}$, let $\bar{\sigma}^{\bar{g}}$ denote the conjugate representation $\bar{\sigma}^{\bar{g}}(\bar{s}) = \bar{\sigma}(\bar{g}\bar{s}\bar{g}^{-1})$, and $m(\bar{\sigma}^{\bar{g}})$ its multiplicity in the space of cusp forms on $\bar{S}_{\mathbf{A}}$. Then*

$$m(\bar{\sigma}^{\bar{g}}) = 1 \quad (\textit{as opposed to } 0)$$

for all $\bar{g}$ in $G_F\bar{Z}_{\mathbf{A}}\bar{S}_{\mathbf{A}}$.

PROOF. Suppose first that $\bar{g} = \bar{z}\bar{s}$ in $\bar{Z}_{\mathbf{A}}\bar{S}_{\mathbf{A}}$. Then $\bar{\sigma}^{\bar{g}}$ is equivalent to $\bar{\sigma}$ (with intertwining operator $\bar{\sigma}(\bar{s})$) and $m(\bar{\sigma}^{\bar{g}}) = m(\bar{\sigma}) = 1$.

Now suppose g is in G_F. Pick μ automorphic on $\bar{Z}_{\mathbb{A}}$ and compatible with $\bar{\sigma}$ on the center $\bar{Z}_{\mathbb{A}} \cap \bar{S}_{\mathbb{A}}$. Because

$$(\bar{\sigma} \times \mu)^g = (\bar{\sigma} \times \mu)^{\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)}, \qquad \text{with } a = \det g \text{ in } F^x,$$

Proposition 1.2.6 implies $(\bar{\sigma} \times \mu)^g$ (as well as $\bar{\sigma} \times \mu$) is automorphic cuspidal. But if $(\bar{\sigma} \times \mu)^g$ is realizable in the space of cusp forms $\{f(\bar{g}), g \in G^*_{\mathbb{A}}\}$, the space of restrictions of these functions to $\bar{S}_{\mathbb{A}}$ then realizes $\bar{\sigma}^g$ as an automorphic cuspidal representation of $\bar{S}_{\mathbb{A}}$. Hence $m(\bar{\sigma}^g) = 1$.

REMARK. We suspect that this result is best possible, i.e.,

$$m(\bar{\sigma}^{\bar{g}}) = 0$$

as soon as $\bar{g}$ lies outside $G_F \bar{Z}_{\mathbb{A}} \bar{S}_{\mathbb{A}}$.

(1.4) *Strong Multiplicity One for* $\bar{S}_{\mathbb{A}}$

The derivation of a strong multiplicity one result for $\bar{S}_{\mathbb{A}}$ using Flicker's analogous result for $\bar{G}_{\mathbb{A}}$ is only slightly less trivial than the multiplicity one result just described.

THEOREM 1.4. (cf. [18], theorem 3, p. 67). *Suppose $\bar{\sigma}$ and $\bar{\sigma}'$ are automorphic cuspidal representations of $\bar{S}_{\mathbb{A}}$ which have the same central character and agree almost everywhere, i.e., for all v outside of a finite set of places S, $\bar{\sigma}_v \approx \bar{\sigma}'_v$. Then $\bar{\sigma}$ and $\bar{\sigma}'$ agree (i.e., are equivalent) everywhere.*

PROOF. Choose a character μ of $\bar{Z}_{\mathbb{A}}/Z_F$ compatible with the central character of $\bar{\sigma}$ (and hence $\bar{\sigma}'$, by assumption). The representations $\bar{\sigma} \times \mu$ and $\bar{\sigma}' \times \mu$ of $\bar{G}^*_{\mathbb{A}}$ are automorphic cuspidal, they agree locally for all $v \notin S$, and their restrictions to $\bar{Z}_{\mathbb{A}}$ agree everywhere. Consider the representations

$$\bar{\pi} = \operatorname{Ind}(\bar{\sigma} \times \mu, \bar{G}^*_{\mathbb{A}}, \bar{G}_{\mathbb{A}}) \quad \text{and} \quad \bar{\pi}' = \operatorname{Ind}(\bar{\sigma}' \times \mu, \bar{G}^*_{\mathbb{A}}, \bar{G}_{\mathbb{A}})$$

of $\bar{G}_{\mathbb{A}}$. They are automorphic cuspidal, they agree almost everywhere, and their central characters (their restrictions to $\bar{Z}^2_{\mathbb{A}}$) agree everywhere. Hence by Flicker's strong multiplicity one theorem for $\bar{G}_{\mathbb{A}}$ ([1], corollary 5.3, p. 180), $\bar{\pi} \approx \bar{\pi}'$. In particular, by multiplicity one, $\bar{\pi}$ and $\bar{\pi}'$ both act in one and the same subspace of cusp forms, call it $V_{\bar{\pi}}$, and the space consisting of the restriction of functions in $V_{\bar{\pi}}$ to $\bar{G}^*_{\mathbb{A}}$ contains both $\bar{\sigma}' \times \mu$ as well as $\bar{\sigma} \times \mu$. By Proposition 1.2.6 (ii) this means

$$\bar{\sigma}' \times \mu \approx (\bar{\sigma} \times \mu)^a = \bar{\sigma}^a \times \mu^a$$

for some a in F^x. (By abuse of notation we write $\bar{\sigma}^a$ for $\bar{\sigma}\left[\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right]$.) But $\bar{\sigma}' \times \mu \approx \bar{\sigma}^a \times \mu^a$ implies $\mu^a = \mu$, an impossibility unless a is a *square* in F^x. Thus $\left[\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right] \in \bar{S}_{\mathbf{A}}\bar{Z}_{\mathbf{A}}$, and

$$\bar{\sigma}' \approx \bar{\sigma}^a \approx \bar{\sigma},$$

as was to be shown.

REMARK 1.4.1. Both the proof and the statement of Theorem 1.4 fail with $S_{\mathbf{A}}$ in place of $\bar{S}_{\mathbf{A}}$. Indeed, as already remarked after Proposition 1.1.4, cuspidal representations of $G^*_{\mathbf{A}}$ don't induce irreducibly to $G_{\mathbf{A}}$, and as shown in [9], strong multiplicity one for $\mathrm{SL}_2(\mathbf{A})$ fails. Here is a simple counterexample.

If $F = \mathbf{Q}$, let π denote an irreducible cuspidal representation of $\mathrm{SL}_2(\mathbf{A})$ whose infinite component π_∞ is a discrete series representation of lowest weight k. For the matrix $\varepsilon = \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ in $\mathrm{GL}_2(F)$, the conjugate representation π^ε is again automorphic cuspidal. Suppose π_p is an irreducible principal series representation of $\mathrm{SL}_2(\mathbf{Q}_p)$ for each p outside some finite set S. Then $\pi_p^\varepsilon \approx \pi_p$ for all $p \notin S$; moreover, the central characters of π_p and π_p^ε agree for *all* p. Nevertheless, π_∞^ε is a discrete series representation of highest weight $-k$. In particular, π^ε is not equivalent to π, contradicting strong multiplicity one.

CONCLUDING REMARK. "Strong Multiplicity One" fails for $\bar{S}_{\mathbf{A}}$ as well *if* we drop our assumption on the central characters. This fact is central (pun intended) to our discussion of "near-equivalence" of cuspidal representations of $\bar{S}_{\mathbf{A}}$ in §4, and was first noticed by Waldspurger (who in [18] produced the interesting and non-trivial counterexamples discussed in §4).

§2. The relation between the correspondences of Shimura and Shintani–Waldspurger

Our goal in this section is to discuss the commutative diagram

$$\begin{array}{ccc} \pi & \xleftarrow{\quad \mathrm{SC} \quad} & \bar{\pi} \\ {\scriptstyle \otimes\omega^{-1}}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}_\mu} \\ \pi_\psi & \xrightarrow{\quad \Theta(\psi) \quad} & \bar{\sigma} \end{array} \tag{2.0}$$

Here $\bar{\pi}$ (resp. $\bar{\sigma}$) is an irreducible genuine representation of $\bar{G}_{\mathbf{A}}$ (resp. $\bar{S}_{\mathbf{A}}$), and π_ψ (resp. π) is an irreducible representation of $\mathrm{PGL}_2(\mathbf{A})$ (resp. $G_{\mathbf{A}}$); SC denotes the (generalized) Shimura correspondence constructed in [3] or [1], Res_μ

denotes the μ-restriction operator described in Proposition 1.3.1, and $\Theta(\psi)$ denotes the Shintani–Waldspurger correspondence mapping genuine representations of $\bar{S}$ possessing a ψ-Whittaker model to irreducible representations of PGL_2; $\omega = \omega_\psi = \mu\chi_{\psi^{-1}}$ is a character of $\mathbf{A}^\times$ whose precise definition will appear below (cf. (2.3.2)).

Not surprisingly, most of our discussion is purely local. First we shall show that the Shimura-type zetafunctions of [3], suitably modified with $\bar{S}$ in place of $\bar{G}$, lead to precisely the correspondence of Waldspurger. In other words, the Shimura correspondence constructed in [3] and the Waldspurger correspondence described in [16] amount to the same construction. From this we are able to give a direct (and local) derivation of all the ε-factor assertions of §6 of [3] as well as the commutativity of our "Key Diagram" (2.0).

Finally, in §3, we discuss global applications of our Key Diagram to the non-vanishing of Hecke's L-functions $L(\pi \otimes \chi_\xi, s)$ at $s = 1/2$, results originally proved by Waldspurger in [17].

(2.1) *Local Shimura-type Integrals on $\bar{S}$*

Throughout this section, F is a local field, and $\bar{\sigma}$ is an irreducible admissible genuine representation of $\bar{S} = \mathrm{SL}_2(F)$. Suppose ψ is a non-trivial additive character of F such that the ψ-Whittaker model $\mathcal{W}(\psi, \bar{\sigma})$ exists. By ψ^ξ we denote the character $\psi^\xi(x) = \psi(\xi x)$, ξ in $F^\times$. Following [3] and [16–18], we let r_ψ denote the "basic Weil representation" of $\bar{S}$ acting in the Schwartz–Bruhat space $\mathcal{S}(F)$. By $\bar{W}_\psi(\bar{h})$ we denote the complex conjugate of any non-zero function in the ψ-Whittaker model of (the even or odd irreducible piece of) r_ψ.

Using the methods of [3], we shall attach to $\bar{\sigma}$ certain L and ε factors which turn out to be exactly the L and ε factors associated (á là Jacquet–Langlands) to the image π_ψ of $\bar{\sigma}$ under $\Theta(\psi)$.

Let $W(\bar{h})$ denote any non-zero function in $\mathcal{W}(\psi, \bar{\sigma})$. For each Φ in the Schwartz-Bruhat space $\mathcal{S}(F \times F)$, each s in $\mathbf{C}$, each quasicharacter α of $F^\times$, and $\bar{h} = (h, 1)$ in $\bar{S}$, put

$$^*f^\Phi(h, s, \alpha) = \int_{F^\times} \Phi((0, a)h) |a|^{s+1/2} \alpha(a) d^\times a$$

and

$$\Psi(s, W, W_\psi, \Phi) = \int_{N\backslash \mathrm{SL}_2(F)} {}^*f^\Phi(h, s, \alpha) W_\psi(h) W(\bar{h}) dh.$$

We note that:

(1) These integrals converge for $\mathrm{Re}(s)$ sufficiently large.

(2) The integrand defining $\Psi(s, W, W_\psi, \Phi)$ depends only on $N \backslash \mathrm{SL}_2(F)$, since W and $\bar{W}_\psi$ transform contravariantly under $n \times \{\pm 1\}$, and N fixes $(0, a)$.

(3) The central characters of the even and odd pieces of r_ψ differ by sign. Hence, having fixed $\tilde{\sigma}$ and α, only one choice of component will produce a non-vanishing integral for $\Psi(s, W, W_\psi, \Phi)$. This is the choice we make.

For any function F in the PGL_2 Whittaker model $\mathcal{W}(\pi_\psi, \psi)$, and $\mathrm{Re}(s)$ sufficiently large, let $\mathrm{JL}(F, s, \alpha)$ denote the Jacquet–Langlands zeta-integral

$$\mathrm{JL}(F, s, \alpha) = \int_{F^\times} F\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} |a|^{s-1/2} \alpha(a) d^\times a;$$

cf. [8], p. 75. The proposition below provides the sought-after relation between the L and ε construction of Shimura-type and those of Jacquet–Langlands.

Proposition 2.1. *For F appropriately related to W, W_ψ, and Φ,*

$$\mathrm{JL}(F, s, \alpha) = \Psi(s, W, W_\psi, \Phi).$$

To prove this proposition we shall exploit a convenient formula for $F(g)$ in $\mathcal{W}(\pi_\psi, \psi)$. But first we shall review Waldspurger's correspondence using the dual reductive pair language of [7].

Let V denote the space of 2×2 trace zero matrices over F equipped with the quadratic form $q(x) = -\det(x)$. If $\mathrm{GL}_2(F)$ acts on V by $(g)(x) = gxg^{-1}$ it is well-known that this action identifies PGL_2 with $\mathrm{SO}(V)$, the special orthogonal group of V.

Let W_1 denote the two-dimensional vector over F and $\langle \cdot, \cdot \rangle$ the standard skew symmetric bilinear form on W_1. Then the isometry group $\mathrm{Sp}(W_1)$ is just $\mathrm{SL}_2(F)$.

To bring into play Howe's theory of dual reductive pairs, form the six-dimensional space $W = V \otimes W_1$, equipped with the skew symmetric form obtained by tensoring the above forms on V and W_1. The groups PGL_2 and SL_2 embed naturally in the isometry group $\mathrm{Sp}(W)$, and the oscillator (or Weil) representation of $\mathrm{Sp}(W)$ produces a (genuine) representation $\omega_\psi(g, h)$ of $\mathrm{PGL}_2 \times \bar{S}$.

To continue, we need an explicit realization of $\omega_\psi(g, h)$. In particular, we need to fix a particular "Schrödinger model" for $\omega_\psi(g, h)$. For this, we choose a basis $\{x_1, l, x_2\}$ for V with

$$x_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad l = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

and a basis $\{w_1, w_2\}$ for W_1 with $w_1 = (1,0)$ and $w_2 = (0,1)$. (The matrix of the form $q(X)$ with respect to the basis $\{x_1, l, x_2\}$ is

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

and the matrix of the form $\langle \cdot \rangle$ on W_1 is $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$.) Define two subspaces of $W = V \otimes W_1$ by

$$Z_1 = \{x_1\} \otimes W_1 \oplus \{l \otimes w_1\} \quad \text{and} \quad Z_1 = \{x_2\} \otimes W_1 \oplus \{l \otimes w_2\}.$$

These subspaces are isotropic for the skew symmetric form on W, and their direct sum is W. In other words, these subspaces comprise a "complete polarization" for W and hence provide a realization of $\omega_\psi(g, \bar{h})$ in $\mathcal{S}(Z_1)$; cf. [7].

For convenience, we write an arbitrary point Z_1 as $z = (w, t)$, with w in W_1 and t in F such that $z = x_1 \otimes w + t(l \otimes w_1)$.

LEMMA 2.1.1. *Suppose $\bar{\sigma}$, ψ and $W(\bar{h})$ are as above, i.e., $\mathcal{W}(\bar{\sigma}, \psi)$ exists and $W(\bar{h})$ is non-zero in $\mathcal{W}(\bar{\sigma}, \psi)$. Suppose ϕ is in $\mathcal{S}(Z_1)$, and put*

$$F(g) = \int_{N \backslash \mathrm{SL}_2(F)} \overline{\omega_\psi(g, \bar{h})\phi(z_1)W(\bar{h})}dh,$$

where $z_1 = (w_2, -1)$ in Z_1, and $g \in \mathrm{PGL}_2(F)$. Then $F(g)$ defines a ψ-Whittaker function for the irreducible representation π_ψ of $\mathrm{PGL}_2(F)$. (Here $\pi_\psi = \Theta(\psi)(\bar{\sigma})$, the image of $\bar{\sigma}$ under the Shintani–Waldspurger correspondence $\Theta(\psi)$.)

PROOF. According to §IV of [18], the space of right translates of $F(g)$ realizes an irreducible representation of $\mathrm{PGL}_2(F)$, namely $\Theta(\psi)(\bar{\sigma})$. Indeed, once Z_1 is properly identified with the space F^3, our operators $\omega_\psi(g, h)$ on $\mathcal{S}(Z_1)$ coincide with the operators $\tilde{r}_\psi(\sigma)\tilde{R}(g)$ on $\mathcal{S}(F^3)$ which Waldspurger defines on page 22 of his manuscript. In particular,

$$\omega_\psi\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} g, \bar{h}\right)\phi(z_1) = \psi(-x)\omega_\psi(g, \bar{h})\phi(z_1),$$

and therefore $F[\left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right]g] = \psi(x)F(g)$, i.e., $F(g)$ is indeed a ψ-Whittaker function.

To prove Proposition 2.1 we shall combine Lemma 2.1.2 with the following observation about how $\omega_\psi(g, \bar{h})$ factors over the subgroup $\{\left[\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right]\} \times \bar{S}$.

The matrices $\left[\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $w = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ in $\mathrm{PGL}_2(F)$ stabilize the vector l (at least up to sign). Hence these matrices act as $O(1)$ in the one-dimensional subspace $\{l\}$ of V. Simultaneously, these matrices act as $O(2)$ in the two-dimensional complementary subspace of V spanned by x_1 and x_2, taking x_1 to ax_1 and x_2 to $a^{-1}x_2$.

Let ω_ψ^1 (resp. ω_ψ^2) denote the Weil representation of $O_1 \times \bar{S}$ (resp. $O_2 \times \bar{S}$) associated to the polarized symplectic space $\{l \otimes w_1\} \oplus \{l \otimes w_2\}$ (resp. $\{x_1\} \otimes W_1 \oplus \{x_2\} \otimes W_1$). The restriction of $\omega_\psi(g, \bar{h})$ to the subgroup generated by the matrices $\left[\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ (together with $\bar{S}$) is the product of the Weil representations ω_ψ^2 and ω_ψ^1 acting in $\mathcal{S}(\{x_1\} \otimes W_1)$ and $\mathcal{S}(\{l\} \otimes w_1)$ respectively.

We can (and do) assume that ϕ in $\mathcal{S}(Z_1)$ is of the form

$$\phi(w, t) = \phi_2(w)\phi_1(t)$$

since such functions span a dense subspace of $\mathcal{S}(Z_1)$. Thus

$$\omega_\psi\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}, \bar{h}\right) \phi(w, t) = \omega_\psi^2\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \bar{h}\right) \phi_2(w) \omega_\psi\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}, \bar{h}\right) \phi_1(t).$$

This factorization of $\omega_\psi(g, \bar{h})$ vindicates our particular choice of polarization (Z_1, Z_2) for W.

How do these representations ω_ψ^1 and ω_ψ^1 act in their respective function spaces? Since $\left[\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right]$ acts trivially on l, $\omega_\psi^1(\left[\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right], \bar{h})\phi_1(t) = r_\psi(\bar{h})\phi_1(t)$, with r_ψ the basic Weil representation in $\mathcal{S}(F)$. In particular, the function $r_\psi(\bar{h})\phi_1(-1)$ is the complex conjugate of a function $W_\psi(\bar{h})$ in the Whittaker space for (the even or odd part of) r_ψ.

On the other hand, the polarization $\{x_1\} \otimes W_1$ for $(O_2, \bar{S})$ "linearizes" the action of $\bar{h}$ in $\omega_\psi^2(g, \bar{h})$, so we have

$$\omega_\psi^2\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}, h\right) \phi_2(w_2) = \phi_2(w_2 a h) \,|\, a \,|.$$

In particular,

$$\int_{F^\times} \overline{\omega_\psi^2\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}, \bar{h}\right) \phi_2(w_2)} \,|\, a \,|^{s-1/2} \alpha(a) d^\times a = \int \overline{\phi_2((0, a)h)} \,|\, a \,|^{s+1/2} \alpha(a) d^\times a$$
$$= {}^*f^{\phi_2}(h, s, \alpha).$$

Thus, with $\Phi = \bar{\phi}_2$ and F, W_ψ and W as above, Lemma 2.1.1 implies $\mathrm{JL}(F, s, \alpha)$

$$= \int_{F^\times} F\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} |\, a \,|^{s-1/2} \alpha(a) d^\times a$$

$$= \int_{F^\times} |\, a \,|^{s-1/2} \alpha(a)$$

$$\times \int_{N \backslash \mathrm{SL}_2(F)} \overline{\omega_\psi^2\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}, \bar{h}\right) \phi_2(w_2) \omega_\psi^1\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}, \bar{h}\right) \phi_1(-1)} W(\bar{h}) dh$$

$$= \int_{N \backslash \mathrm{SL}_2(F)} {}^*f^{\Phi}(h, s, \alpha) \bar{W}_\psi(\bar{h}) W(\bar{h}) dh = \Phi(s, W, W_\psi, \Phi),$$

as was to be shown.

Proposition 2.1 is now proved; we note that similar arguments show that (with $w = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$)

$$\begin{aligned} \mathrm{JL}(wF, 1-s, \alpha^{-1}) &= \int_{F^\times} F\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} w\right) |a|^{3/2-s} \alpha^{-1}(a) d^\times a \\ &= \int_{N\backslash \mathrm{SL}_2(F)} {}^*f^\Phi(h, 1-s, \alpha^{-1}) W(\bar{h}) \overline{W_\psi(\bar{h})} dh \\ &= \tilde{\Psi}(1-s, W, W_\psi, \hat{\Phi}) \end{aligned}$$

with

$$\hat{\Phi}(x, y) = \int_{F \times F} \Phi(u, v) \psi(xv - yu) du dv$$

and F, W, W_ψ and Φ as above. But according to the local functional equation of Jacquet–Langlands (cf. [8], p. 75),

$$\frac{\mathrm{JL}(wF, 1-s, \alpha^{-1})}{L(1-s, \tilde{\pi}_\psi \alpha^{-1})} = \varepsilon(\pi_\psi \otimes \alpha, s, \psi) \frac{\mathrm{JL}(F, s, \alpha)}{L(s, \pi_\psi \otimes \alpha)} \tag{2.1.2}$$

for any Whittaker function $F(g)$ in $\mathcal{W}(\pi_\psi, \psi)$, and any quasicharacter α of $F^\times$. (Here $L(s, \pi_\psi \otimes \alpha)$ and $\varepsilon(\pi_\psi \otimes \alpha, s, \psi)$ are the familiar functions of Jacquet–Langlands.) Thus, by Proposition 2.1 and the above remarks, we also have

$$\frac{\tilde{\Psi}(1-s, W, W_\psi, \hat{\Phi})}{L(1-s, \tilde{\pi}_\psi \otimes \alpha^{-1})} = \varepsilon(s, \pi_\psi \otimes \alpha, \psi) \frac{\Psi(s, W, W_\psi, \Phi)}{L(s, \pi_\psi \otimes \alpha)}$$

for all W, W_ψ and Φ. This equation is the $\bar{S}$-analogue of the local "Shimura-type" functional equation for $\overline{\mathrm{GL}_2}(F)$ developed in [3]. It implies that the L and ε factors attached to Shimura-type integrals for $\bar{\sigma}$ are just $L(s, \pi_\psi \otimes \alpha)$ and $\varepsilon(s, \pi_\psi \otimes \alpha, \psi)$. Globally, similar reasoning relates $\mathrm{JL}(F, s, \alpha)$ to the (convergent) integral of the product of a cusp form for $\bar{\sigma}$, a theta-series, and an Eisenstein series. This makes it possible to prove (by now familiar reasoning) that $L(s, \pi_\psi \otimes \alpha)$ is entire and satisfies the kind of functional equation necessary to show that π_ψ is indeed an automorphic cuspidal representation of $\mathrm{PGL}_2(\mathbf{A})$ (at least when $\bar{\sigma}$ is not itself a "basic theta-series" r_ν; cf. the first paragraph of §2.2).

Summing Up. The natural $\bar{S}$-adaptation of the Shimura-type L-function constructions of [3] yields precisely the correspondence $\Theta(\psi)$ of Waldspurger.

2.2. *The ε-factors of [3] Revisited*

First we recall how these factors are defined. If $\bar{\pi}$ is any genuine irreducible admissible representation of $\bar{G}$ (F is still local), we pick μ to be any genuine character of $\bar{Z}$ which extends the central character of $\bar{\pi}$ and is such that the (ψ, μ)-Whittaker model $\mathcal{W}(\bar{\pi}, \psi, \mu)$ of [6] exists. If χ is any character of $F^{\times}$, we let r_{χ} denote the irreducible basic Weil representation of $\overline{\mathrm{GL}_2}(F)$ described in [4]. This is a "distinguished" representation; given ψ, $\mathcal{W}(r_{\chi}, \psi, \mu) \neq \{0\}$ if and only if $\mu(a) = \chi(a)\chi_{\psi}(a)$. Here χ_{ψ} is a certain projective character of $F^{\times}$ whose restriction to $(F^{\times})^2$ is trivial and whose precise definition appears in equation (1.2.2) of [4] or p. 4 of [16]. For each non-zero $W(\bar{g})$ in $\mathcal{W}(\bar{\pi}, \psi, \mu)$, $W_{\chi}(\bar{g})$ in $\mathcal{W}(r_{\chi}, \psi^{-1}, \chi_{\psi^{-1}}\chi)$ and Φ in $\mathcal{S}(F \times F)$, we define the zeta-integrals

$$\Psi(s, W, W_{\chi}, \Phi) = \int_{NZ \backslash \mathrm{GL}_2(F)} W(g) W_{\chi}(g) f^{\Phi}(g, s, \omega_*) dg$$

and

$$\tilde{\Psi}(s, W, W_{\chi}, \Phi) = \int_{NZ \backslash G} W(g) W_{\chi}(g) \alpha^{-1}(\det g) f^{\Phi}(g, s, \omega_*^{-1}) dg,$$

with

$$f^{\Phi}(g, s, \omega_*) = |\det g|^s \int_{F^{\times}} \Phi((0 \;\; a)g) |a|^{2s} \omega_*(a) da,$$

and

$$\omega_* = \chi \mu \chi_{\psi}^{-1}.$$

These functions extend meromorphically to all of $\mathbf{C}$, with g.c.d.'s $L(s, \bar{\pi}, \chi)$ and $\tilde{L}(s, \bar{\pi}, \chi)$ respectively. Moreover (cf. theorem 5.3 of [3]), they satisfy the functional equation

$$\frac{\tilde{\Psi}(1-s, W, W_{\chi}, \hat{\Phi})}{\tilde{L}(1-s, \bar{\pi}, \chi)} = \frac{\varepsilon(s, \bar{\pi}, \chi, \psi) \Psi(s, W, W_{\chi}, \Phi)}{L(s, \bar{\pi}, x)}. \tag{2.2.1}$$

To compute these L-factors we analysed the asymptotic behaviour of the Whittaker functions $W\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $W_{\chi}\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Since the asymptotic behavior of these functions is independent of the choice of μ (as well as ψ), so are the factors $L(s, \bar{\pi}, \chi)$ and $\tilde{L}(s, \bar{\pi}, \chi^{-1})$; these results are described in §6 of [3].

Our purpose here is to explain an efficient method of computation for the ε-factors $\varepsilon(s, \bar{\pi}, \chi, \psi)$. The fact that the functional equation (2.2.1) holds for all choices of W, W_{χ} and Φ means that the ε-factor can be computed using a special (judicious) choice of these functions. In the present context this amounts to the following.

Regarding $\mathcal{W}(\bar{\mu}, \psi, \mu)$ as in $\bar{G}$-module, we know from §1.1 that

$$\mathcal{W}(\bar{\pi}, \psi, \mu) = \bigoplus_{\mu \in \Omega_{\bar{\pi}}} \mathcal{W}^{\mu}$$

with $\mathcal{W}^{\mu}$ the subspace of $W(g)$ in $\mathcal{W}(\bar{\pi}, \psi, \mu)$ such that $W(g\bar{z}) = \mu(\bar{z})W(g)$ for all $\bar{z}$ in $\bar{Z}$. Recall that each $\mathcal{W}^{\mu}$ realizes an irreducible representation $\bar{\pi}^{\mu}$ of $\bar{G}^*$ with central character μ. Now choose $W(g)$ in $\mathcal{W}^{\mu}$. Then

$$W(\tilde{g}) = \frac{W(g\bar{z})}{\mu(z)} = (\det g, z)\frac{W(\bar{z}g)}{\mu(z)}$$

$$= (\det g, z)W(\bar{g}),$$

i.e., $W(\bar{g})$ vanishes off the set $\bar{G}^*$ (where $(\det g, z) \equiv 1$).

With this choice of $W(g)$,

$$\Psi(s, W, W_{\chi}, \Phi) = \int_{NZ \backslash G^*} W(g)W_{\chi}(g)f^{\Phi}(g, s, \omega_*)dg$$

$$= \int_{N \backslash \mathrm{SL}_2} W(h)W_{\psi^{-1}}(h)^* f^{\Phi}(h, 2s - 1/2, \alpha)dg$$

where $W(h)$ is a function in the ψ-Whittaker model of the irreducible representation $\bar{\sigma}$ of $\bar{S}$ obtained by restricting $\bar{\pi}^{\mu}$ to $\bar{S}$, W_{ψ} is a function in the ψ^{-1}-Whittaker space of an irreducible piece of $r_{\psi^{-1}}$ (even or odd depending on whether $\chi(-1) = 1$ or -1), and $\alpha = \omega_* = \mu\chi_{\psi^{-1}}\chi$.

Now assume $W_{\chi}(g)$ chosen so that $W_{\psi^{-1}}(\bar{h})$ is of the form $r_{\psi^{-1}}(\bar{h})\phi_1(1)$, with ϕ_1 in $\mathcal{S}(F)$. Thus, since

$$r_{\psi^{-1}}(\bar{h})\phi_1(t) = \overline{r_{\psi}(\bar{h})\phi_1(t)}$$

we conclude that $W_{\psi^{-1}}(\bar{h})$ is then the complex conjugate of a ψ-Whittaker function for (the even or odd part of) r_{ψ}. In other words, if $s' = 2s - 1/2$, then $\Psi(s, W, W_{\chi}, \Phi)$ coincides with the integral $\Psi(s', W, W_{\psi^{-1}}, \Phi)$ just described for $\mathrm{SL}_2(F)$. A similar identity holds for $\tilde{\Psi}(1-s, W, W_{\chi}, \tilde{\Psi})$.

What does this imply about the factor $\varepsilon(s, \bar{\pi}, \chi, \psi)$? If $\bar{\pi}$ is a supercuspidal representation of $\bar{G}$ (and not of the form r_{ν} for any ν on $F^{\times}$) then $L(s, \bar{\pi}, \chi) = 1$. Thus

$$\varepsilon(s, \bar{\pi}, \chi, \psi) = \frac{\Psi(1-s, W, W_{\psi^{-1}}, \hat{\Phi})}{\Psi(s, W, W_{\psi^{-1}}, \Phi)},$$

with the integrals on the right those defined on $SL_2(F)$, and W in $\mathcal{W}^{\mu}$ so chosen that $\Psi(s, W, W_\chi, \Phi) \neq 0$. In other words, by Proposition 2.1 and (2.1.2),

$$\varepsilon(s, \bar{\pi}, \chi, \psi) = \varepsilon(s, \pi_\psi \otimes \alpha, \psi).$$

On the one hand, this identity gives us the desired computation of $\varepsilon(s, \bar{\pi}, \chi, \psi)$. At the same time, it shows that $\varepsilon(s, \bar{\pi}, \chi, \psi)$ is independent of the choice of μ in $\Omega_{\bar{\pi}}$.

Indeed, suppose $\mu^* \neq \mu$ is another such genuine character of $\bar{Z}$. Then for some ξ in $F^\times$, $\mu^*(z) = \chi_\xi(z)\mu(z)$ with χ_ξ the quadratic character $\chi_\xi(x) = (\xi, x)$; i.e., choosing $W^*(g)$ in $\mathcal{W}^{\mu^*}$ in place of $W(g)$ in $\mathcal{W}^{\mu}$ leads to the character $\alpha^*(x) = \alpha(x)\chi_\xi(x)$ in place of α.

On the other hand, the representation of $\bar{\pi}^{\mu^*}$ is conjugate to π^{μ} via the element $\left[\begin{smallmatrix} \xi & 0 \\ 0 & 1 \end{smallmatrix}\right]$, and the restriction of $W^*(\bar{g})$ to $\bar{S}$ is conjugate to $W(\bar{h})$, i.e.,

$$W^*(\bar{h}) = W(\bar{h}^{\left[\begin{smallmatrix} \xi & 0 \\ 0 & 1 \end{smallmatrix}\right]}).$$

So from Lemma 2.1.1 together with the fact that

$$\omega_\psi(g, h^{\left[\begin{smallmatrix} \xi & 0 \\ 0 & 1 \end{smallmatrix}\right]}) = \omega_{\psi^\xi}(g, h),$$

it follows that the resulting Whittaker function $F^*(g)$ on PGL_2 belongs to the representation π_{ψ^ξ} instead of π_ψ. But $\pi_{\psi^\xi} = \pi_\psi \otimes \chi_\xi$. Thus this different choice of μ, namely μ^*, leads to the factor

$$\begin{aligned}
\varepsilon(s, \bar{\pi}, \chi, \psi, \mu^*) &= \varepsilon(s', \pi_{\psi^\xi} \otimes \alpha^*, \psi) \\
&= \varepsilon(s', \pi_\psi \otimes \chi_\xi \otimes \alpha \otimes \chi_\xi, \psi) \\
&= \varepsilon(s', \pi_\psi \otimes \alpha, \psi) = \varepsilon(s, \bar{\pi}, \chi, \psi, \mu)
\end{aligned}$$

as was to be shown.

A similar argument works when $\bar{\pi}$ is not supercuspidal (or of the form r_ν with $\nu(-1) = -1$). In this case, $L(s, \bar{\pi}, \chi)$ and $\tilde{L}(x, \bar{\pi}, \chi)$ are no longer always 1, but the L-function computations of §6 of [3] show that

$$L(s, \bar{\pi}, \chi) = L(s', \pi_\psi \otimes \alpha) \quad \text{and} \quad \tilde{L}(s, \bar{\pi}, \chi) = L(s', \pi_\psi \otimes \alpha^{-1}).$$

Here π_ψ is the representation of $PGL_2(F)$ which Waldspurger's ψ-correspondence attaches to the restriction of the "μ-component" of $\bar{\pi}$ to $\bar{S}$, and $\alpha = \omega_* = \mu\chi\chi_{\psi^{-1}}$. Thus, choosing $W(g)$ as above, the functional equation reads

$$\begin{aligned}
\varepsilon(s, \bar{\pi}, \chi, \psi) &= \frac{\Psi(1-s', W, W_\psi, \hat{\Phi})L(s', \pi_\psi \otimes \alpha)}{\Psi(s', W, W_\psi, \Phi)L(s', \bar{\pi}_\psi \otimes \alpha^{-1})} \\
&= \varepsilon(s', \pi_\psi \otimes \alpha, \psi),
\end{aligned}$$

as was to be shown.

Summing Up. The ε-factor $\varepsilon(s,\bar{\pi},\chi,\psi,\mu)$ is indeed independent of the choice of μ (cf. §5.4 of [3]). Moreover, consider the diagram

(2.2.2)
$$\begin{tikzcd} \pi \arrow[d, "?"'] & & \bar{\pi} \arrow[ll, "\mathrm{SC}"'] \arrow[d, "\mathrm{Res}_{\mu}(\bar{\pi})"] \\ \pi_{\psi} & & \bar{\sigma} \arrow[ll] \end{tikzcd}$$

(with μ such that $\mathscr{W}(\bar{\pi},\psi,\mu)$ exists). Then for any quasicharacter χ of F^{x},

$$\varepsilon(s,\bar{\pi},\chi,\psi)=\varepsilon(s',\pi_{\psi}\otimes\alpha,\psi),$$

with $\alpha=\mu\chi_{\psi^{-1}}\chi$, and $s'=2s-1/2$. From this, the ε-factor assertions of sections 6 and 7 of [3] follow.

(2.3) *Shimura's Correspondence and the Key Diagram*

F is still local.

In §7.1 of [3] we defined an irreducible admissible representation π of G to be the *Shimura image of $\bar{\pi}$ on $\bar{G}$* if

(2.3.1)
$$\omega\pi\begin{pmatrix} a & 0\\ 0 & a\end{pmatrix}=\omega_{\bar{\pi}}\begin{pmatrix} a^2 & 0\\ 0 & a^2\end{pmatrix}$$

and, for any quasicharacter χ of F^{x},

$$L(s,\bar{\pi},\chi)=L(s',\pi\otimes\chi),$$

$$\tilde{L}(s,\bar{\pi},\chi)=L(s',\tilde{\pi}\otimes\chi^{-1})$$

and

$$\varepsilon(s,\bar{\pi},\chi,\psi)=\varepsilon(s',\pi\otimes\chi,\psi).$$

Since the (twisted) L and ε factors π uniquely determine π (subject to (2.3.1)), such a Shimura image — denoted by $\mathrm{SC}(\bar{\pi})$ — is unique. What we have just shown is that this Shimura correspondence is such that for all quasicharacters χ of F^{x},

$$L(s',\pi\otimes\chi)=L(\pi_{\psi}\otimes\alpha,s')=L(s,\bar{\pi},\chi),$$

and

$$\varepsilon(s',\pi\otimes\chi,\psi)=\varepsilon(s,\pi_{\psi}\otimes\alpha,s')=\varepsilon(s,\bar{\pi},\chi,\psi),$$

with $\alpha=\mu\chi_{\psi^{-1}}\chi$. Thus

$$\pi\otimes\chi=\pi_{\psi}\otimes\alpha,$$

and hence

$$\pi = \pi_\psi \otimes \alpha\chi^{-1};$$

i.e., our diagram (2.2.2) commutes, with the operation ? defined by tensoring by ω^{-1}, and ω defined by the equation

$$\omega = \alpha\chi^{-1} = \mu\chi_{\psi^{-1}}. \tag{2.3.2}$$

§3. Global applications of the Key Diagram

If $\bar{\pi} = \bigotimes \bar{\pi}_v$ is an irreducible admissible genuine representation of $\bar{G}_{\mathbf{A}}$ with central character $\omega_{\bar{\pi}}$, and $\pi = \bigotimes \pi_v$ is an irreducible admissible representation of $G_{\mathbf{A}}$, we say π is the Shimura image of $\bar{\pi}$, and write $\pi = \mathrm{SC}(\bar{\pi})$, if each $\pi_v = \mathrm{SC}(\bar{\pi}_v)$. In [3] we proved that every genuine cuspidal representation $\bar{\pi}$ has a Shimura image, and this image is itself automorphic cuspidal (provided $\bar{\pi}$ is not distinguished in the sense of [4], i.e., a basic Weil representation of the form r_ν). Moreover, Flicker subsequently proved in [1] that $\pi = \bigotimes \pi_v$ is in the image of SC if and only if it satisfies the following two conditions.

(F1) The central character of each π_v, regarded as a character of $F_v^\times$, is the square of a character of $F_v^\times$; equivalently, $\omega_{\pi_v}(-1) = 1$;

(F2) Whenever π_v is equivalent to a principal series representation $\pi_v(\mu_1, \mu_2)$, *both* μ_1 and μ_2 are "even", i.e., squares of characters.

Using Flicker's theorem, we can now give a simple proof of one of Waldspurger's most beautiful results (see the end of [17]).

THEOREM 3.1. *Suppose $\pi = \bigotimes \pi_v$ is an automorphic cuspidal representation of* $\mathrm{PGL}_2(\mathbf{A})$ *such that at least one π_v belongs to the discrete series. (For example, this condition holds whenever $F = \mathbf{Q}$ and π corresponds to a classical modular cusp form of weight k.) Then there exists a ξ in $F^\times$ such that $L(\pi \otimes \chi_\xi, \frac{1}{2}) \neq 0$.*

PROOF. The first step is to produce a grossencharacter χ such that $\pi \otimes \chi$, regarded as a representation of $\mathrm{GL}_2(\mathbf{A})$, satisfies Flicker's conditions (F1) and (F2).

Let S be the set of places v of F such that π_v is a principal series representation $\pi_v,(\nu, \nu^{-1})$ with ν *odd* instead of even. Whenever π_v is unramified, $v \notin S$. Thus S is at worst finite. If S is empty, π already satisfies condition (F1) and (F2), and we can take $\chi \equiv 1$. In general, if the cardinality of S is even, let $\chi = \Pi_v \chi_v$ be a grossencharacter such that $\chi_v(-1) = -1$ for v in S, and $\chi_v(-1) = 1$ for $v \notin S$. If the cardinality of S is odd, let χ be such that $\chi_v(-1) = -1$ if and only if $v \in S \cup \{v_0\}$. (Here v_0 is any place such that π_{v_0}

belongs to the discrete series.) Then $\pi \otimes \chi$ satisfies both of Flicker's conditions, and hence is in the image of Shimura's correspondence SC. (Cf. the example on p. 13 of [15].)

Let $\bar{\pi}$ be the automorphic cuspidal representation of $\bar{G}_{\mathbf{A}}$ corresponding to $\pi \otimes \chi$ via Shimura's correspondence. If μ is a (genuine) character of $\bar{Z}_{\mathbf{A}}/Z_F$ extending $\omega_{\bar{\pi}}$, let $\bar{\sigma}$ be the cuspidal representation of $\bar{S}_{\mathbf{A}}$ obtained by restriction to $V_{\bar{\pi}}\mu$. Choose ψ' on $\mathbf{A}/F$ such that $\mathcal{W}(\bar{\sigma}, \psi') \neq \{0\}$, and let π' be the image of $\bar{\sigma}$ under the correspondence $\Theta(\psi')$. In diagram language:

$$\begin{tikzcd} \pi \otimes \chi & \bar{\pi} \arrow[l, "SC"'] \arrow[d, "\mathrm{Res}_{|V_{\bar{\pi}}\mu}"] \\ \pi' & \bar{\sigma} \arrow[l, "\Theta(\psi')"'] \end{tikzcd}$$

By the commutativity of our Diagram (2.0), $\pi' \otimes \chi' = \pi \otimes \chi$ with $\chi' = \bar{\chi}_{\psi'}\mu$. This implies $\pi' = \pi \otimes \chi''$, with $\chi'' = \chi(\chi')^{-1}$. But the restriction of χ' to $(\mathbf{A}^{\times})^2$ agrees with $\omega_{\bar{\pi}}$ on $(\mathbf{A}^{\times})^2$, and $\omega_{\bar{\pi}}$ in turn coincides with χ on $(\mathbf{A}^{\times})^2$. Thus χ'' defines a character of $F^{\times}\ \mathbf{A}^{\times}/(\mathbf{A}^{\times})^2$, which means $\chi'' = \chi_{\xi}$ for some ξ in $F^{\times}$.

On the other hand, by Waldspurger's characterization of the image of $\Theta(\psi')$ ([16], prop. 27 and theorem 1), $L(\pi, \frac{1}{2}) \neq 0$. So since $\pi' = \pi \otimes \chi'' = \pi \otimes \chi_{\xi}$, we have $L(\pi \otimes \chi_{\xi}, \frac{1}{2}) \neq 0$ and the theorem is proved.

(3.2) Our second application is to adapt Waldspurger's criterion for the non-vanishing of metaplectic Fourier coefficients from $\bar{S}_{\mathbf{A}}$ to $\bar{G}_{\mathbf{A}}$. For this we need to review some additional notation from [4].

If $\bar{\pi}$ is any genuine cuspidal representation of $\bar{G}_{\mathbf{A}}$, and ψ is a fixed non-trivial character of $\mathbf{A}/F$, let $\Omega_{\bar{\pi}}(\psi)$ denote the set of characters μ of $\bar{Z}_{\mathbf{A}}/Z_F$ such that the restriction of μ to $\bar{Z}^2_{\mathbf{A}}$ agrees with $\omega_{\bar{\pi}}$ and such that a (ψ, μ)-Whittaker model for $\bar{\pi}$ exists. For each $\delta \in F^{\times}$, the (ψ^{δ}, μ)-Whittaker space $\mathcal{W}(\bar{\pi}, \psi^{\delta}, \mu)$ is generated by the (ψ^{δ}, μ)-Fourier coefficients

$$W^{\phi}_{\psi^{\delta},\mu}(\bar{g}) = \int_{\bar{Z}^2_{\mathbf{A}} \backslash \bar{Z}_{\mathbf{A}}} \int_{F \backslash \mathbf{A}} \phi\left(\bar{z} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \bar{g}\right) \psi^{-1}(\delta x) \mu^{-1}(\bar{z}) dx d\bar{z}$$

of cusp forms $\phi(\bar{g})$ in the space $V_{\bar{\pi}}$. The Fourier expansion of $\phi(\bar{g})$ is given in terms of these $W^{\phi}_{\psi^{\delta},\mu}(\bar{g})$ by the formula

(3.2.1) $$\phi(\bar{g}) = \sum_{\delta \in F^{\times}} \sum_{\mu \in \Omega_{\bar{\pi}}(\psi)} W^{\phi}_{\psi^{\delta},\mu}(\bar{g}).$$

In general, ϕ is not completely determined by any "first Fourier coefficient" $W_{\psi,\mu_0}(\bar{g})$. If it is, we call ϕ (and $\bar{\pi}$) *distinguished*. In this case, $W^{\phi}_{\psi,\mu}(\bar{g}) \neq \{0\}$ for only one choice of automorphic μ in $\Omega_\pi(\psi)$, say μ_0, and the Fourier expansion (3.2.1) reduces to the simpler (more familiar) formula

$$\phi(\bar{g}) = \sum_{\delta \in F^\times} W^{\phi}_{\psi,\mu_0}\left(\begin{bmatrix} \delta & 0 \\ 0 & 1 \end{bmatrix} \bar{g}\right).$$

In [4] we proved that the distinguished cusp forms ϕ are precisely those generated by theta series attached to quadratic forms in one variable. In other words, for the remaining generic $\bar{\pi}$ now considered, it is *never* true that all but one "orbit" of Fourier coefficients of ϕ vanish identically. Indeed, the subtle nature of the vanishing of an arbitrary Fourier coefficient $W^{\phi}_{\psi,\mu}(\bar{g})$ is related à la Waldspurger to the non-vanishing of an appropriate L-function $L(\pi', s)$ at $s = 1/2$!

Waldspurger's result is the following (cf. [18]). Suppose $\bar{\sigma} = \bigotimes \bar{\sigma}_v$ is a cuspidal automorphic representation of $\bar{S}_{\mathbf{A}}$, and ψ is such that that non-zero $\mathcal{W}(\bar{\sigma}, \psi)$ exists. (In this case $\pi' = \Theta(\psi)(\bar{\sigma})$ is non-zero by prop. 26 of [16].) Then for any ξ in $F^\times$, $\mathcal{W}(\bar{\sigma}, \psi^\xi)$ is non-zero if and only if the following two conditions are satisfied:

(i) $L(\pi' \otimes \chi_\xi, \frac{1}{2}) \neq 0$, and

(ii) $\mathcal{W}(\bar{\sigma}_v, \psi_v^{\xi_v})$ exists for each v.

For the next theorem, suppose $\bar{\pi}$ is a (non-distinguished) cuspidal representation of $\bar{G}_{\mathbf{A}}$. For any non-trivial character ψ' of $\mathbf{A}/F$, and any μ in $\Omega_{\bar{\pi}}$ as above, let $\mathcal{W}(\bar{\pi}, \psi', \mu)$ denote the space of (ψ', μ)-Fourier coefficients of cusp forms ϕ in the space $V_{\bar{\pi}}$ of $\bar{\pi}$. Let χ denote the grossencharacter of F defined by the formula $\chi = \bar{\chi}_{\psi'}\mu$, and put $\pi = \mathrm{SC}(\bar{\pi})$.

THEOREM 3.2. *$\mathcal{W}(\bar{\pi}, \mu, \psi') \neq \{0\}$ iff $L(\pi \otimes \chi^{-1}, \frac{1}{2}) = 0$ and for each place v there exists a (ψ'_v, μ_v)-Whittaker functional on the space of $\bar{\pi}_v$.*

PROOF. Because multiplicity one holds for the space of cusp forms on $\bar{G}_{\mathbf{A}}$ ([1], p. 180), $V_{\bar{\pi}}$ is "the" subspace of cusp forms realizing $\bar{\pi}$, and $\mathcal{W}(\bar{\pi}, \mu, \psi)$ depends only on the class of $\bar{\pi}$.

Let us suppose first that $\mathcal{W}(\bar{\pi}, \psi', \mu) \neq \{0\}$. With this hypothesis we want to produce a cuspidal representation $\bar{\sigma}$ of $\bar{S}_{\mathbf{A}}$ such that $\mathcal{W}(\bar{\sigma}, \psi') \neq \{0\}$, and the corresponding representation $\Theta(\psi')(\bar{\sigma})$ is just $\pi \otimes \chi^{-1}$.

As in §1, consider the decomposition

$$V_{\bar{\pi}_\bullet} = \bigoplus V^{\mu}_{\bar{\pi}_\bullet}.$$

The action of $\bar{S}_{\mathbf{A}}$ in $V^{\mu}_{\bar{\pi}_\bullet}$ defines an automorphic cuspidal representation of $\bar{S}_{\mathbf{A}}$ which we shall call $\bar{\sigma}$.

Let l denote the non-trivial (ψ',μ)-Whittaker functional defined on $V_{\bar\pi}$ by evaluation of $W^{\phi}_{\psi',\mu}$ at the identity e. Regarding l as a functional on $V_{\bar\pi,}$, we claim l is non-trivial on the subspace $V^{\mu}_{\bar\pi,}$. Indeed if v belongs to $V^{\mu'}_{\bar\pi_\bullet}$, $\mu'\neq\mu$, then

$$\mu'(\bar z)l(v)=l(\pi(\bar z)v)=\mu(\bar z)l(v),\qquad \text{for } z\in \bar Z_{\mathbf{A}},$$

and this implies $l(v)=0$. Thus l vanishes identically on $(V^{\mu}_{\bar\pi_\bullet})^{\perp}$, and hence cannot vanish on $V^{\mu}_{\bar\pi,}$.

On $V^{\mu}_{\bar\pi_\bullet}$ we have

$$l(f)=\int\int f\left(\bar z\begin{bmatrix}1 & x\\ 0 & 1\end{bmatrix}\bar e\right)\psi'(-x)\mu^{-1}(\bar z)dxd\bar z$$

$$=\int_{\mathbf{A}/F} f\left(\begin{bmatrix}1 & x\\ 0 & 1\end{bmatrix}\bar e\right)\psi'(-x)dx = W^{f}_{\psi'}(\bar e).$$

Thus $\mathcal{W}(\bar\sigma,\psi')\neq\{0\}$, and the Shintani–Waldspurger ψ'-image of $\bar\sigma$ exists.

Now let $\pi'=\Theta(\psi')(\bar\sigma)$ and consider the diagram

$$\begin{array}{ccc}\pi & \xleftarrow{\quad\text{SC}\quad} & \bar\pi\\ & & \big\downarrow \text{Res}_{\mu}\\ \pi' & \longleftarrow & \bar\sigma\end{array}$$

By the commutativity of our diagram (2.0), $\pi'=\pi\otimes\chi^{-1}$, with $\chi=\bar\chi_{\psi'}\mu$. Therefore $L(\pi',\frac12)\neq 0$ really says $L(\pi\otimes\chi^{-1},\frac12)\neq 0$, as was to be shown. The existence of a local (ψ'_v,μ_v)-Whittaker model for $\bar\pi_v$ is of course an immediate consequence of the assumed non-vanishing of $\mathcal{W}(\bar\pi,\psi',\mu)$.

Conversely, suppose $L(\pi\otimes\chi^{-1},\frac12)\neq 0$, and $\mathcal{W}(\bar\pi_v,\psi'_v,\mu_v)$ exists for each v. Let $\bar\sigma$ still denote the irreducible cuspidal representation of $\bar S_{\mathbf{A}}$ derived from $\bar\pi$ by "π-restriction". Because $\mathcal{W}(\bar\pi_v,\psi'_v,\mu_v)$ exists for each v it is clear from the local theory of §1.1 that $\mathcal{W}(\bar\sigma_v,\psi'_v)$ exists for each v.

Now suppose ψ_0 is a character of $F\backslash\mathbf{A}$ such that $\mathcal{W}(\bar\sigma,\psi_0)\neq 0$ and suppose $\psi'=(\psi_0)^{\xi}$. Let π^0 denote the $\Theta(\psi_0)$ image of $\bar\sigma$. By our key diagram, $\pi^0=\pi\otimes\chi^{-1}\chi_{\xi}$. Therefore, since $\mathcal{W}(\bar\sigma_v,(\psi_{0,v})^{\xi_v})$ exists for each v, and $L(\pi^0\otimes\chi_{\xi},\frac12)=L(\pi\otimes\chi^{-1},\frac12)\neq 0$, the hypotheses of Waldspurger's $\bar S_{\mathbf{A}}$ result are satisfied. We conclude then that $\mathcal{W}(\bar\sigma,\psi')\neq\{0\}$, and hence — by reversing the argument used in the first part of this proof — that $\mathcal{W}(\bar\pi,\psi',\mu)\neq\{0\}$.

Concluding Remark. Though the proof of Theorem 3.2 *assumes* multiplicity one for $\bar G_{\mathbf{A}}$, the theorem itself also implies it. Indeed, the condition $L(\pi\otimes\chi^{-1},\frac12)\neq 0$ is independent of the imbedding of $\bar\pi$ into the space of cusp forms, and the (ψ,μ)-models $\mathcal{W}(\bar\pi,\psi,\mu)$, unique by [6], uniquely determine the subspace. Thus it would be of interest to give a direct proof of the theorem which avoids the work of Waldspurger or Flicker. We believe such a proof is possible using Rankin–Selberg–Shimura integrals of the type investigated in [3] and §2.1.

§4. Near-equivalence of metaplectic cusp forms

Following [7], we shall call two genuine representations of $\bar{S}_{\mathbf{A}}$ *near-equivalent* if they are equivalent at almost all local places v. Given an automorphic cuspidal genuine representation $\bar{\sigma} = \bigotimes \bar{\sigma}_v$ of $\bar{S}_{\mathbf{A}}$, a natural problem is to describe the size and nature of the intersection of its near-equivalence class with the set of *all* genuine cuspidal representations of $\bar{S}_{\mathbf{A}}$.

In the first draft of this paper (November 1981), we obtained some preliminary results on near-equivalence "packets" and presented some conjectures about their precise description. Subsequently, J. L. Waldspurger proved these conjectures (and much more) in the preprint [18].

Because Waldspurger himself makes reference to this portion of our original manuscript, and because we feel this section still contributes some useful insights, we include (an updated version of) it herewith.

4.1. *General Remarks*

Suppose $\bar{\sigma}$ and $\bar{\sigma}'$ are near-equivalent irreducible cuspidal genuine representations of $\bar{S}_{\mathbf{A}}$. Then $\bar{\sigma}$ and $\bar{\sigma}'$ need not be equivalent everywhere, despite our strong multiplicity one result (Theorem 1.4). Indeed, $\bar{\sigma}$ will be equivalent to $\bar{\sigma}'$ if and only if their central characters coincide everywhere.

Before describing the near equivalence class of $\bar{\sigma}$, let us specify a useful modification of the Shintani–Waldspurger correspondence $\Theta(\psi)$. For an arbitrary non-trivial character ψ of $\mathbf{A}/F$, the correspondence $\Theta(\psi)$ is well-defined on $\bar{\sigma}$ if and only if the Whittaker model $\mathcal{W}(\psi, \bar{\sigma})$ exists. Moreover, on the set of such $\bar{\sigma}$, $\Theta(\psi)$ is a bijection onto the set of irreducible cuspidal representations of $\pi = \bigotimes \pi_v$ of $\mathrm{PGL}_2(\mathbf{A})$ such that $L(\pi, \frac{1}{2}) \neq 0$, and this bijection has a local definition as well.

Now drop the assumption that $\mathcal{W}(\psi, \bar{\sigma})$ exist. Picking $\xi \in F^x$ such that $\mathcal{W}(\psi^{\xi}, \bar{\sigma})$ *does* exist, Waldspurger defines a representation $\pi = S_{\psi}(\bar{\sigma})$ of $\mathrm{PGL}_2(\mathbf{A})$ by the formula

$$S_{\psi}(\bar{\sigma}) = \Theta(\psi^{\xi})(\bar{\sigma}) \otimes \chi_{\xi}.$$

See [18], p. 66; according to prop. 28 of [16], this correspondence S_{ψ} is well-defined, i.e., independent of the choice of $\xi \in F^\times$ such that $\mathcal{W}(\psi^\xi, \bar\sigma)$ exists. In particular, if $\mathcal{W}(\psi, \bar\sigma)$ exists, then $S_\psi(\bar\sigma) = \Theta(\psi)(\bar\sigma)$.

Henceforth, we fix ψ once and for all.

PROPOSITION 4.1. *Suppose $\bar\sigma$ and $\bar\sigma'$ are near-equivalent cuspidal representations of $\bar S_{\mathbf{A}}$ as above, and let π* (resp. π') *denote $S_\psi(\bar\sigma)$* (resp. $S_\psi(\bar\sigma')$). *Then for some ξ in $F^\times$,*

$$\bar\sigma' = \Theta(\psi^\xi)^{-1}(\pi \otimes \chi_\xi).$$

PROOF. By definition, $\pi = \Theta(\psi^\lambda)(\bar\sigma) \otimes \chi_\lambda$, and $\pi' = \Theta(\psi^{\lambda'})(\bar\sigma') \otimes \chi_{\lambda'}$, with λ (resp. λ') in $F^\times$ such that $\mathcal{W}(\psi^\lambda, \bar\sigma)$ (resp. $\mathcal{W}(\psi^{\lambda'}, \bar\sigma')$) exists. Since $\bar\sigma$ and $\bar\sigma'$ are assumed to agree almost everywhere, so must the cuspidal representations π and π' agree almost everywhere. Indeed, for almost all v, $\bar\sigma_v$ and $\bar\sigma'_v$ will be equivalent *principal* series representation, and for such representations,

$$\Theta(\psi_v^{\lambda_v})(\bar\sigma_v) \otimes \chi_{\lambda_v} \approx \Theta(\psi_v)(\bar\sigma_v) \approx \Theta(\psi_v^{\lambda'_v})(\bar\sigma'_v) \otimes \chi_{\lambda'_v};$$

cf. prop. 18 of [16]. Thus, by strong multiplicity one for PGL_2, π and π' are actually equivalent, i.e.,

$$\pi = \Theta(\psi^{\lambda'})(\bar\sigma') \otimes \chi_{\lambda'}.$$

Tensoring both sides of this equation by χ'_λ gives

$$\pi \otimes \chi_{\lambda'} = \Theta(\psi^{\lambda'})(\bar\sigma'),$$

and applying $\Theta(\psi^\lambda)^{-1}$ to both sides of this last equation gives

$$\bar\sigma' = \Theta(\psi^{\lambda'})^{-1}(\pi \otimes \chi_{\lambda'}).$$

Taking $\xi = \lambda'$ gives exactly what was to be proved.

COROLLARY 4.1.1. *Each automorphic cuspidal genuine near-equivalence class on $\bar S_{\mathbf{A}}$ is of the form*

$$\mathrm{NE}(\pi) = \{\Theta(\psi^\xi)^{-1}(\pi \otimes \chi_\xi) : \xi \in F^\times\}$$

where π is some automorphic cuspidal representation of $\mathrm{PGL}_2(\mathbf{A})$ *with the property that $L(\pi \otimes \chi_\xi, \frac{1}{2}) \neq 0$ for some ξ in $F^\times$.*

PROOF. By the proposition, any two near-equivalent cuspidal $\bar\sigma$ and $\bar\sigma'$ belong to one fixed set of the form $\mathrm{NE}(\pi)$. Conversely, if π is as above, then the family of cuspidal representations $\Theta(\psi^\xi)^{-1}(\pi \otimes \chi_\xi)$, $\xi \in F^\times$, are near-equivalent. Indeed, as remarked in the first part of the proof of the proposition,

$\Theta(\psi_v^{\xi_v})^{-1}(\pi_v \otimes \chi_{\xi_v})$ is independent of ξ_v whenever π_v is a principal series representation.

COROLLARY 4.1.2. *Let $\bar{\sigma}$ denote any genuine cuspidal automorphic representation of $\bar{S}_{\mathbf{A}}$ which is not a basic Weil representation of the form r_ν. Let* NE $(\bar{\sigma})$ *denote the set of all cuspidal automorphic representations which are near-equivalent to $\bar{\sigma}$. Then* NE $(\bar{\sigma})$ *is a finite set.*

PROOF. By Corollary 4.1.1, NE $(\bar{\sigma})$ is of the form NE (π). Let S denote the (finite) set of places where π_v belongs to the discrete series for $\mathrm{PGL}_2(F_v)$. By our assumption that $\bar{\sigma} \not\approx r_v$ it follows that $\bar{\sigma}_v(\xi_v) = \Theta(\psi_v^{\xi_v})^{-1}(\pi_v \otimes \chi_{\xi_v})$ is a principal series representation of $\bar{S}_v$ for all v outside S. In particular, $\bar{\sigma}_v(\xi_v)$ is independent of ξ for all v outside S.

On the other hand, for v inside S, $\bar{\sigma}_v(\xi_v)$ — regarded as a function on $F_v^\times$ — is at least constant on cosets of $F_v^\times$ modulo $(F_v^\times)^2$. Indeed

$$\begin{aligned}\bar{\sigma}_v(b) = \Theta(\psi_v^b)^{-1}(\pi_v \otimes \chi_b) &= [\Theta(\psi_v)^{-1}(\pi_v)]^\varepsilon \\ &= \Theta(\psi_v)^{-1}(\pi_v) = \bar{\sigma}_v(1),\end{aligned}$$

where $b = a^2$ and $\varepsilon = \left[\begin{smallmatrix} b & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Therefore, since S is finite, and the cardinality of $F_v^\times/(F_v^\times)^2$ is finite, so is NE (π) = NE $(\bar{\sigma})$.

REMARK 4.1.3. Proposition 4.1 suggests the following *local* definition of near-equivalence. If $\bar{\sigma}$ is a genuine irreducible admissible representation of $\bar{S}$ which corresponds to π on $\mathrm{PGL}_2(F)$, then $\bar{\sigma}'$ is *near-equivalent* to $\bar{\sigma}$ if and only if $\bar{\sigma}' = \bar{\sigma}(\xi) = \Theta(\psi_v)^{-1}(\pi \otimes \chi_\xi)$ for some ξ in $F^\times$. The proof of Corollary 4.1.2 shows that NE $(\bar{\sigma}) = \{\bar{\sigma}(\xi) : \xi \in F^\times\}$ is a finite set, in fact a singleton set whenever $\bar{\sigma}_v$ is not a discrete series representation.

REMARK 4.1.4. When $\bar{\sigma}$ is a cuspidal representation of the form r_ν ("distinguished" in the sense of [4]), NE$(\bar{\sigma})$ is *infinite.*

COROLLARY 4.1.5. *Suppose $\bar{\sigma}$ and $\bar{\sigma}'$ are near equivalent, but not of the form r_ν. Let Σ* (resp. Σ') *denote the (finite) set of places v of F where $\bar{\sigma}_v$* (resp. $\bar{\sigma}'_v$) *is square-integrable. Then $\Sigma = \Sigma'$, and $\bar{\sigma}'_v \approx \bar{\sigma}_v$ for all $v \notin S$.*

PROOF. Obvious from the above discussion.

4.2. *Near-Equivalence and Quaternion Algebras*

Let D denote a non-split quaternion algebra central over F (local or global). By Howe's philosophy of dual reductive pairs, it is natural to use Weil's representation to obtain a correspondence $\Theta_D(\psi)$ between (irreducible or automorphic) representations of $\bar{S}$ and the projective group of $D^\times$ analogous to the correspondence $\Theta(\psi)$. The result is a diagram

$$\begin{tikzcd} (\bar{\sigma}, \bar{S}) \arrow[rr, leftrightarrow, "\Theta_D(\psi)"] \arrow[dr, leftrightarrow] & & (\pi, \mathrm{PGL}_2) \arrow[dl, leftrightarrow] \\ & (\pi', PD^\times) & \end{tikzcd} \tag{4.2.1}$$

where JL denotes the bijection between representations of GL_2 and $D^\times$ constructed in §16 of [8]. This is the diagram which Waldspurger has analyzed in great detail — both locally and globally — in [18]. A crucial point is that this diagram is usually *not* commutative. On the one hand, this leads to Waldspurger's "counterexample" to strong multiplicity one for $\bar{S}_{\mathbf{A}}$ when the assumption on central characters is dropped, i.e., to examples of near equivalent cusp forms. On the other hand, it is precisely this phenomenon which makes it possible to give a truly intriguing description of the near equivalence sets $\mathrm{NE}(\bar{\sigma})$.

Here is a reformulation of the local result which we conjectured in the original draft of this paper and which Waldspurger proves in [18].

THEOREM 4.2.2. *Suppose $\bar{\sigma}$ is a discrete series representation of $\bar{S} = \overline{\mathrm{SL}_2}(F)$, and ψ is a non-trivial additive character of F (local) such that the Whittaker model $\mathcal{W}(\psi, \bar{\sigma})$ exists. Then:*

(a) *Those ξ in $F^\times$ such that $\mathcal{W}(\psi^\xi, \bar{\sigma})$ fails to exist fill out exactly half the cosets of $F^\times$ modulo squares; i.e., the near-equivalence set $\mathrm{NE}(\bar{\sigma}) = \{\bar{\sigma}(\xi) = \Theta(\psi^\xi)^{-1}(\pi \otimes \chi_\xi)\}$ consists of precisely two elements;*

(b) *Suppose the two elements of $\mathrm{NE}(\bar{\sigma})$ are $\bar{\sigma} = \Theta(\psi)^{-1}(\pi)$ and $\bar{\sigma}^* = \bar{\sigma}(\xi)$. Then $\bar{\sigma}^* = \Theta(\psi^\xi)^{-1}(\pi \otimes \chi_\xi) = \Theta_D(\psi)^{-1}(\pi')$, where π' is the discrete series representation $\mathrm{JL}(\pi)$ on $D^\times$ (and D is the unique division quaternion algebra defined over F).*

REMARK. We may think of the map

$$\bar{\sigma} \to \bar{\sigma}^*$$

as an involution which naturally partitions the genuine discrete series representations of $\bar{S}$ into, pairs of representations which coincide with the (square-integrable) near-equivalence packets $N(\bar{\sigma})$. In case $F = \mathbf{R}$ and $\bar{\sigma}$ is a "holomorphic discrete series representation of lowest weight $k/2$", $\bar{\sigma}^*$ is the corresponding "anti-holomorphic discrete series representation of highest weight $-k/2$". In this case, $\bar{\sigma}^*$ is actually the conjugate (cf. (1.1.1)) of $\bar{\sigma}$ by the matrix $\left[\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right]$, but in general $\bar{\sigma}^*$ need have no relation to any conjugate $\bar{\sigma}^g$ of $\bar{\sigma}$.

The global analog of Theorem 4.2.2 is wonderful but now not so terribly surprising (cf. [18]).

THEOREM 4.2.3. *Suppose $\bar{\sigma}$ is a genuine automorphic cuspidal representation of $\bar{S}_{\mathbf{A}}$ not of the form r_{ψ}, and Σ is the (finite) set of places of F where $\bar{\sigma}_v$ is a discrete series representation. Then*

$$|\mathrm{NE}(\bar{\sigma})| = \begin{cases} 2^{|\Sigma|-1} & \text{if } \Sigma \neq \varnothing, \\ 1 & \text{if } \Sigma = \varnothing. \end{cases}$$

More precisely, for each subset T of Σ such that $|T|$ is even, define $\bar{\sigma}^T = \bigotimes \bar{\sigma}_v^T$ by

$$\bar{\sigma}_v^T = \begin{cases} \bar{\sigma}_v^* & \text{if } v \in T, \\ \bar{\sigma}_v & \text{if } v \notin T. \end{cases}$$

Then $T \to \bar{\sigma}^T$ defines a bijection from the set of such even subsets of Σ to the set of $\bar{\sigma}'$ in $\mathrm{NE}(\bar{\sigma})$.

For the proof, see [18].

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